The Core

Game Theory Course:
Jackson, Leyton-Brown & Shoham

Stable payoff division

- The Shapley value defined a fair way of dividing the grand coalition's payment among its members.
  - However, this analysis ignored questions of stability.
- Would the agents be willing to form the grand coalition given the way it will divide payments, or would some of them prefer to form smaller coalitions?
  - Unfortunately, sometimes smaller coalitions can be more attractive for subsets of the agents, even if they lead to lower value overall.

Example: Voting Game

Example (Voting game)

A parliament is made up of four political parties, A, B, C, and D, which have 45, 25, 15, and 15 representatives, respectively. They are to vote on whether to pass a $100 million spending bill and how much of this amount should be controlled by each of the parties. A majority vote, that is, a minimum of 51 votes, is required in order to pass any legislation, and if the bill does not pass then every party gets zero to spend.

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- Shapley values: $(50, 16.67, 16.67, 16.67)$.
- Can a subcoalition gain by defecting? While A can't obtain more than 50 on its own, A and B have incentive to defect and divide the $\$100$ million between them (e.g., $(75, 25)$).

The Core

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- They would want to do so if and only if the payment profile is drawn from a set called the core.

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Definition (Core)

A payoff vector x is in the core of a coalitional game (N, v) iff

$$\forall S \subseteq N, \sum_{i \in S} x_i \geq v(S).$$

- The sum of payoffs to the agents in any subcoalition S is at least as much as they could earn on their own.
- Analogous to Nash equilibrium, except that it allows deviations by groups of agents.

Existence and Uniqueness

1. Is the core always nonempty?

2. Is the core always unique?

Existence and Uniqueness

1. Is the core always nonempty? No.
   - Consider again the voting game.
   - The set of minimal coalitions that meet the required 51 votes is $\{A, B\}$, $\{A, C\}$, $\{A, D\}$, and $\{B, C, D\}$.
   - If the sum of the payoffs to parties B, C, and D is less than $100 million, then this set of agents has incentive to deviate.
   - If B, C, and D get the entire payoff of $100 million, then A will receive $0 and will have incentive to form a coalition with whichever of B, C, and D obtained the smallest payoff.
   - Thus, the core is empty for this game.

2. Is the core always unique?

Existence and Uniqueness

1. Is the core always nonempty?

2. Is the core always unique? No.
   - Consider changing the example so that an 80% majority is required
   - The minimal winning coalitions are now $\{A, B, C\}$ and $\{A, B, D\}$.
   - Any complete distribution of the \$100 million among A and B now belongs to the core
     - all winning coalitions need the support of these two parties.

Simple Games

Definition (Simple game)

A game $G = (N, v)$ is **simple** if for all $S \subset N$, $v(S) \in \{0, 1\}$.

Definition (Veto player)

A player i is a **veto player** if $v(N \setminus \{i\}) = 0$.

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Theorem

In a simple game the core is empty iff there is no veto player. If there are veto players, the core consists of all payoff vectors in which the nonveto players get 0.

Airport Game

Example (Airport game)

Several nearby cities need airport capacity, with different cities needing to accommodate aircraft of different sizes. If a new regional airport is built the cities will have to share its cost, which will depend on the largest aircraft that the runway can accommodate. Otherwise each city will have to build its own airport.

This situation can be modeled as a coalitional game (N, v), where N is the set of cities, and $v(S)$ is the sum of the costs of building runways for each city in S minus the cost of the largest runway required by any city in S.

Convex games

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A game $G = (N, v)$ is convex if for all $S, T \subset N$,
$v(S \cup T) \geq v(S) + v(T) - v(S \cap T)$.

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- Convexity is a stronger condition than superadditivity.
- The Airport game is convex.

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Theorem

Every convex game has a nonempty core.

Theorem

In every convex game, the Shapley value is in the core.